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THE METHOD OF LIAPUNOV FUNCTIONS IN THE STABILITY PROBLEM FOR MOTION WITH RESPECT TO A PART OF THE VARIABLES

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Liapunov ([1], p. 370) posed the problem of the stability of motion with respect to a part of the variables. Malkin [2] in his remarks on Liapunov's theorems stated (without proof) certain conditions for carrying Liapunov's theorems over to this problem. The concept of a function $V(t, x_1, \ldots, x_n)$ which is sign-definite relative to $x_1, \ldots, x_m$ $(m \leqslant n)$ was introduced in [3] and related theorems were proved generalizing Liapunov's theorems [1]. Subsequently there appeared a number of papers [4 - 15] which validated the possibility of applying the theorems of Liapunov's second method (and their modifications and generalizations) for this specific problem. A number of surveys [16 - 19] exist in the literature on the stability theory of motion, however, there is a lack of surveys on stability investigations with respect to a part of the variables. The present paper is an attempt at filling this gap and gives a survey of the results in this area obtained to date. In the paper we have introduced a unified notation and uniform formulations which do not always coincide in form with those of the original authors but which do completely reflect their sense.

1. Basic definitions. We consider a system of differential equations of perturbed motion

$$x_i^{\cdot} = X_i(t, x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

or, in vector form,

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}), \qquad \mathbf{X}(t, 0) \equiv 0 \tag{1.1}$$

We concern ourselves with the question of the stability of the unperturbed motion $\mathbf{x} = 0$ with respect to a part of the variables, to be specific, with respect to $x_1, \ldots, x_m$ $(m > 0$, $n = m + p$, $p \geqslant 0)$. For brevity we denote these variables by $y = x_i$ $(i = 1, \ldots, m)$ and the rest by $z_j = x_{m+j}$ $(j = 1, \ldots, n - m = p)$, i.e., $\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p)$. We introduce the notation

$$\|\mathbf{y}\| = \Big(\sum_{i=1}^{m} y_i^2\Big)^{1/2}, \quad \|\mathbf{z}\| = \Big(\sum_{j=1}^{p} z_j^2\Big)^{1/2}, \quad \|\mathbf{x}\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} = (\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2)^{1/2}$$

We assume that:

(a) in the region

$$t \geqslant 0, \qquad \|\mathbf{y}\| \leqslant H > 0, \qquad 0 \leqslant \|\mathbf{z}\| < +\infty \tag{1.2}$$

the right-hand sides of system (1.1) are continuous and satisfy the conditions for uniqueness of the solution;

(b) the solutions of system (1.1) are $\mathbf{z}$-extendable; this means [6] that any solution

$\mathbf{x}(t)$ is defined for all $t \geqslant 0$ for which $\|\mathbf{y}(t)\| \leqslant H$.

By $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ we denote the solution of system (1.1) determined by the initial conditions $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$.

Definition 1. The motion $\mathbf{x} = 0$ is said to be:

(a) stable relative to $x_1, \ldots, x_m$ [1], or y-stable, if for any $\varepsilon > 0$, $t_0 \geqslant 0$, no matter how small ε is, we can find $\delta(\varepsilon, t_0) > 0$ such that for every $t \geqslant t$, $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ follows from $\|\mathbf{x}_0\| < \delta$;

(b) y-stable uniformly in t_0 [5, 6, 9] it in Definition 1a we can choose $\delta(\varepsilon)$ independently of t_0 for each $\varepsilon > 0$;

(c) asymptotically y-stable [3, 5, 6, 9] if it is y-stable and for every $t_0 \geqslant 0$ there exists $\Delta(t_0) > 0$ such that the solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ with $\|\mathbf{x}_0\| < \Delta$ satisfies the condition

$$\lim_{t\to\infty} \|\mathbf{y}(t; t_0, \mathbf{x}_0)\| = 0 \tag{1.3}$$

(here we say that the region $\|\mathbf{x}\| < \Delta$ lies in the region of y-attraction of the point $\mathbf{x} = 0$ for the initial instant t_0);

(d) asymptotically y-stable uniformly in $\{t_0, \mathbf{x}_0\}$ [5, 6, 9] if it is y-stable uniformly in t_0 and there exists a number $\Delta_0 > 0$ independent of t_0 such that condition (1.3) is fulfilled uniformly with respect to $\{t_0, \mathbf{x}_0\}$ from the region

$$t_0 \geqslant 0, \qquad \|\mathbf{x}_0\| < \Delta_0$$

i.e. for any $\varepsilon > 0$ we can find $T(\varepsilon) > 0$ such that for all $t \geqslant t_0 + T$, $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ follows from $t_0 \geqslant 0$, $\|\mathbf{x}_0\| < \Delta_0$.

(e) asymptotically y-stable in-the-large [9] if it is y-stable and condition (1.3) is fulfilled for any $t_0 \geqslant 0$ and $\mathbf{x}_0$, i.e., if the region of y-attraction of the point $\mathbf{x} = 0$ is the whole space; here it is assumed that the right-hand sides of system (1.1) satisfy the Conditions (a) and (b) indicated for them, in the region

$$t \geqslant 0, \qquad 0 \leqslant \|\mathbf{x}\| < +\infty \tag{1.4}$$

(f) exponentially-asymptotically y-stable [11] if there exists constants $M > 0$ and $\alpha > 0$ such that

$$\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \leqslant M(\|\mathbf{y}_0\| + \|\mathbf{z}_0\|)\exp[-\alpha(t - t_0)], \qquad t \geqslant t_0 \geqslant 0 \tag{1.5}$$

We shall be considering certain real single-valued functions $V(t, \mathbf{x})$, continuous and possessing continuous partial derivatives $\partial V / \partial t$, $\partial V / \partial x_i$ $(i = 1, \ldots, n)$ in region (1.2), satisfying the condition $V(t, 0) \equiv 0$, as well as their total time derivative $V^{\cdot}(t, \mathbf{x})$ taken by virtue of system (1.1)

$$V^{\cdot}(t, \mathbf{x}) = \frac{\partial V(t, \mathbf{x})}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, \mathbf{x})}{\partial x_i} X_i(t, \mathbf{x})$$

Definition 2. A function $W(y_1, \ldots, y_m) \equiv W(\mathbf{y})$, not depending explicitly on time, is said to be positive-definite [1] if it is nonnegative in the region $\|y\| \leqslant H$ and vanishes if and only if $y = 0$. A function $V(t, \mathbf{x})$ is said to be y-positive-definite [3] if there exists a positive-definite function $W(\mathbf{y})$, not depending explicitly on t, such that in region (1.2)

$$V(t, \mathbf{x}) \geqslant W(\mathbf{y}) \tag{1.6}$$

Lemma 1. A function $V(t, \mathbf{x})$ is y-positive-definite if and only if there exists a continuous function $a(r)$, monotonically increasing for $r \in [0, H]$ $a(0) = 0$ such that in region (1.2) [5, 6, 9]

$$V(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\|) \tag{1.7}$$

Proof. The sufficiency of inequality (1.7) is obvious. We prove the necessity. We set $b(r) = \min[W(\mathbf{y}): \|\mathbf{y}\| = r]$. Then $V(t, \mathbf{x}) \geqslant b(\|\mathbf{y}\|)$ and, moreover, $b(r)$ is continuous by virtue of the continuity of W. If $b(r)$ increases monotonically on $[0, H]$, then we take $a(r) = b(r)$, otherwise, we can take $a(r) = \varphi(r)\min[b(s): r \leqslant s \leqslant H]$, where $\varphi(r)$ is a function increasing monotonically on $[0, H]$ and, moreover, $0 \leqslant \varphi \leqslant \leqslant 1$. From Lemma 1 it follows that inequalities (1.6) and (1.7) are equivalent.

Definition 3. The function $V(t, \mathbf{x})$ is called y-positive definite in the region (1.4) if (1.6) is fulfilled in the whole of this region and for any $\varepsilon > 0$

$$\inf[W(\mathbf{y}): \varepsilon \leqslant \|\mathbf{y}\| < \infty] > 0$$

This definition is equivalent to (1.7) with monotonically increasing function $a(r)$ for $r \in [0, \infty)$; the proof is analogous to that of Lemma 1.

Definition 4. A function $V(t, \mathbf{x})$ is said to be positive-definite in $x_1, \ldots, x_k$, $m \leqslant \leqslant k \leqslant n$, (for the y-stability problem) if $V(t, \mathbf{x}) \geqslant W(x_1, \ldots, x_k)$ in region (1.2), where $W(0, \ldots, 0) = 0$ and for any $\varepsilon > 0$

$$\inf\Big[W(x_1, \ldots, x_k): \quad \sum_{i=1}^{k} x_i^2 \geqslant \varepsilon^2, \quad \|\mathbf{y}\| \leqslant H\Big] > 0$$

It is not difficult to prove the validity of

Lemma 2. Definition 4 is equivalent to the fulfillment in region (1.2) of the inequality

$$V(t, \mathbf{x}) \geqslant a\bigg(\Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}\bigg)$$

with a continuous function $a(r)$, monotonically increasing for $r \in [0, \infty)$, $a(0) = 0$.

Definition 5. A function $V(t, \mathbf{x})$ admits of an infinitesimal upper bound in $x_1, \ldots, \ldots x_k$, $m \leqslant k \leqslant n$, if for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $|V(t, \mathbf{x})| < \varepsilon$ follows from

$$t \geqslant 0, \quad \sum_{i=1}^{k} x_i^2 < \delta^2, \quad -\infty < x_{k+1}, \ldots, x_n < +\infty$$

This signifies that

$$V(t, \mathbf{x}) \to 0 \qquad \text{as} \quad \sum_{i=1}^{k} x_i^2 \to 0$$

uniformly in $t \geqslant 0$ and $-\infty < x_{k+1}, \ldots, x_n < +\infty$.

Lemma 3. A function $V(t, \mathbf{x})$ admits of an infinitesimal upper bound in y (in $x_1, \ldots, x_k$, $m \leqslant k \leqslant n$) if and only if there exists a function $b(r)$ of the same type as $a(r)$ in Lemma 1, for which in region (1.2) [20]

$$|V(t, \mathbf{x})| \leqslant b(\|\mathbf{y}\|) \tag{1.8}$$

respectively,

$$|V(t, \mathbf{x})| \leqslant b\bigg(\Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}\bigg) \tag{1.9}$$

In particular, $V(t, \mathbf{x})$ admits of an infinitesimal upper bound in x if and only if

$$|V(t, \mathbf{x})| \leqslant b(\|\mathbf{x}\|) \tag{1.10}$$

2. Stability and instability. 1. Theorem 1. (1) If system (1.1) is such that a function $V(t, \mathbf{x})$ satisfying inequality (1.7) exists, while $V^{\cdot} \leqslant 0$, then the motion $\mathbf{x} = 0$ is y-stable [3].

(2) If, furthermore, V satisfies inequality (1.10), then the y-stability is uniform in

t_0 [5].

(3) If the conditions in Sect. 1 are fulfilled and V satisfies inequality (1.8), then for any $\varepsilon > 0$ we can find $\eta(\varepsilon) > 0$ such that from $t_0 \geqslant 0$, $\|\mathbf{y}_0\| < \eta$ $(0 \leqslant \|z_0\| < \infty)$ follows $\|\mathbf{y}(t;\, t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$ [8]; here it is necessary to fulfill the identities [12]

$$X_i(t, 0, \mathbf{z}) \equiv 0 \qquad (i = 1, \ldots, m) \tag{2.1}$$

Proof. (1) For every $\varepsilon > 0$ $(\varepsilon < H)$, $t_0 \geqslant 0$ we can find $\delta(\varepsilon, t_0) > 0$ such that $V(t_0, \mathbf{x}_0) < a(\varepsilon)$ follows from $\|\mathbf{x}_0\| < \delta$. For the solution $\mathbf{x} = \mathbf{x}(t;\, t_0, \mathbf{x}_0)$ with $\|\mathbf{x}_0\| < \delta$ by virtue of $V^{\cdot} \leqslant 0$ from the relation

$$V(t, \mathbf{x}(t;\, t_0, \mathbf{x}_0)) = V(t_0, \mathbf{x}_0) + \int_{t_0}^{t} V^{\cdot}(\tau, \mathbf{x}(\tau;\, t_0, \mathbf{x}_0))\, d\tau \tag{2.2}$$

we obtain $V(t, \mathbf{x}(t;\, t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0)$ for $t \geqslant t_0$. Thus

$$a(\|\mathbf{y}(t;\, t_0, \mathbf{x}_0)\|) \leqslant V(t, \mathbf{x}(t;\, t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) < a(\varepsilon) \tag{2.3}$$

whence $\|\mathbf{y}(t;\, t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$.

(2) When inequality (1.10) is fulfilled we can choose $\delta(\varepsilon) = b^{-1}(a(\varepsilon))$ independent of t_0 (b^{-1} is the inverse of function b). If $\|\mathbf{x}_0\| < \delta$, then

$$V(t_0, \mathbf{x}_0) \leqslant b(\|\mathbf{x}_0\|) < b(b^{-1}(a(\varepsilon))) = a(\varepsilon)$$

(3) For each $\varepsilon > 0$ we set $\eta(\varepsilon) = b^{-1}(a(\varepsilon))$. If $t_0 \geqslant 0$, $\|\mathbf{y}_0\| < \eta$, then $V(t_0, \mathbf{x}_0) < a(\varepsilon)$ and (2.3) holds, whence $\|\mathbf{y}(t;\, t_0, \mathbf{x}_0)\| < \varepsilon$ for $t \geqslant t_0$.

Let us prove the indentities (2.1). We consider the solution $\mathbf{x} = \mathbf{x}(t;\, t_0, 0, \mathbf{z}_0)$ for arbitrary $t_0 \geqslant 0$ and $\mathbf{z}_0$. By virtue of (1.8), $V(t_0, 0, \mathbf{z}_0) = 0$. Since $V \geqslant 0$, while $V^{\cdot} \leqslant 0$, from (2.2) follows $V(t, \mathbf{x}(t;\, t_0, 0, \mathbf{z}_0)) \equiv 0$, whence

$$\|\mathbf{y}(t;\, t_0, 0, \mathbf{z}_0)\| \equiv 0 \tag{2.4}$$

Equality (2.4) is equivalent to identities (2.1) (*). The theorem is proved.

Note. The requirement of an infinitesimal upper bound for the function V implies, as also in the classical case [20], the uniformity of the $\mathbf{y}$-stability. An analogous conclusion obtains also in the case of asymptotic $\mathbf{y}$-stability.

In [3] it was shown that it is possible to apply to the $\mathbf{y}$-stability problem Chetaev's method [21] for constructing the function V in the form of a bundle of integrals of system (1.1).

Theorem 2. [9]. If a function $V(t, \mathbf{x})$ satisfying inequality (1.7) exists and its derivative

$$V^{\cdot}(t, \mathbf{x}) \leqslant -c(\|\mathbf{y}\|) \tag{2.5}$$

$c(r)$ is a function of the type of $a(r)$, then for any $\varepsilon \in (0, H)$, $t_0 \geqslant 0$ we can find $\delta(t_0) > 0$ and $T(t_0, \varepsilon) > 0$ such that for every $\mathbf{x}_0$ with $\|\mathbf{x}_0\| < \delta$ there exists an instant $t_* \in (t_0, t_0 + T)$ for which $\|\mathbf{y}(t_*;\, t_0, \mathbf{x}_0)\| < \varepsilon$.

Proof. According to Theorem 1, (1) for each $t_0 \geqslant 0$ there exists $\delta(t_0) > 0$ such that from $\|\mathbf{x}_0\| < \delta$ follows $\|\mathbf{y}(t;\, t_0, \mathbf{x}_0)\| < H$ for all $t \geqslant t_0$ Let $\lambda(t_0) = \sup [V(t_0, \mathbf{x}) : \|\mathbf{x}\| < \delta]$. We set $T(t_0, \varepsilon) = \lambda(t_0) / c(\varepsilon)$. If $\varepsilon \leqslant \|\mathbf{y}(t;\, t_0, \mathbf{x}_0)\| < H$ for $t \in (t_0, t_0 + T)$, then from (2.2) follows

$$0 < a(\varepsilon) \leqslant V(t_0 + T, \mathbf{x}(t_0 + T;\, t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) - c(\varepsilon) T \leqslant 0$$

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*) We obtain (2.1) by substituting the solution $\mathbf{x} = \mathbf{x}(t;\, t_0, 0, \mathbf{z}_0)$ into system (1.1) and by taking (2.4) and the arbitrariness of $t_0 \geqslant 0$ and of $\mathbf{z}_0$ into account. The converse follows from the uniqueness of the solution.

which is impossible. The theorem is proved.

Let us consider certain generalizations of the results presented.

2. Theorem 3. [5]. (1) If there exists a function $V(t, \mathbf{x})$ possessing the properties:

(a) $V(t, 0) \equiv 0$, $V(t, \mathbf{x})$ is continuous at the point $\mathbf{x} = 0$;

(b) V satisfies inequality (1.7);

(c) $V(t, \mathbf{x}(t; t_0\, \mathbf{x}_0))$ does not increase on any solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ as long as $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \leqslant H$,

then the motion $\mathbf{x} = 0$ is $\mathbf{y}$-stable.

(2) If, furthermore, V satisfies inequality (1.10), then the $\mathbf{y}$-stability is uniform in t_0.

To prove this we note that $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0)$ follows from condition (c). The rest of the proof coincides with the proofs (1) and (2) of Theorem 1. The converse assertion is valid for the second part of the theorem:

Theorem 4 [5]. If the motion $\mathbf{x} = 0$ is $\mathbf{y}$-stable uniformly in t_0, then there exists a function $V(t, \mathbf{x})$ satisfying the conditions in part (2) of Theorem 3.

Proof. We set $V(t, \mathbf{x}) = \sup[\|\mathbf{y}(t + \sigma;\ t, \mathbf{x})\| : \sigma \geqslant 0]$. Obviously, $V(t, \mathbf{x}) \geqslant \|\mathbf{y}\|$ and $V(t, \mathbf{x}) \leqslant \varepsilon(\|\mathbf{x}\|)$ (ε is taken from the definition of uniform $\mathbf{y}$-stability). Further, we have

$$V(t, \mathbf{x}(t;\ t_0, \mathbf{x}_0)) = \sup_{\sigma \geqslant 0} \|\mathbf{y}(t + \sigma;\ t, \mathbf{x}(t;\ t_0, \mathbf{x}_0))\| = \sup_{\sigma \geqslant 0} \|\mathbf{y}(t + \sigma;\ t_0, \mathbf{x}_0)\|$$

Here we have made use of the uniqueness of the solution

$$\mathbf{x}(t;\ \tau, \mathbf{x}(\tau;\ t_0, \mathbf{x}_0)) = \mathbf{x}(t;\ t_0, \mathbf{x}_0) \tag{2.6}$$

Let $t_1 > t_2 \geqslant t_0$, then

$$V(t_1, \mathbf{x}(t_1;\ t_0, \mathbf{x}_0)) = \sup[\|\mathbf{y}(t_1 + \sigma;\ t_0, \mathbf{x}_0)\| : \sigma \geqslant 0] \leqslant$$
$$\leqslant \sup[\|\mathbf{y}(t_2 + \sigma;\ t_0, \mathbf{x}_0)\| : \sigma \geqslant 0] = V(t_2, \mathbf{x}(t_2;\ t_0, \mathbf{x}_0))$$

Thus $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0))$ does not increase. The theorem is proved.

3. A somewhat different approach to the study of $\mathbf{y}$-stability was proposed in [4]. Suppose that we know beforehand the (arbitrarily crude) estimates

$$|z_j(t;\ t_0, \mathbf{x}_0)| \leqslant \eta_j(t;\ t_0, \mathbf{x}_0) A_j(t) \qquad (j = 1, \ldots, p) \tag{2.7}$$

where $\eta_j \to 0$ as $\|\mathbf{x}_0\| \to 0$ uniformly in $t \in [t_0, \infty)$ and the $A_j(t)$ are positive functions continuously differentiable for $t \geqslant 0$. In system (1.1) we make the change of variables

$$\xi_i = x_i, \quad \xi_j A_j = x_j \quad (i = 1, \ldots, m;\ j = m + 1, \ldots, n) \tag{2.8}$$

here the ξ_i satisfy the system of equations

$$\xi_i^{\cdot} = X_i(t, \xi_1, \ldots, \xi_m, \xi_{m+1}A_{m+1}, \ldots, \xi_n A_n) \qquad (i = 1, \ldots, m) \tag{2.9}$$
$$\xi_j^{\cdot} = [-A_j^{\cdot}\xi_j + X_j(t, \xi_1, \ldots, \xi_m, \xi_{m+1}A_{m+1}, \ldots, \xi_n A_n)] / A_j \qquad (j = m + 1, \ldots, n)$$

The following is obvious.

Lemma 4. The $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$ of system (1.1) is equivalent to the Liapunov stability of the motion $\xi = 0$ of system (2.9).

Theorem 5 [4]. For the uniform in t_0 $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$ of system (1.1) it is necessary and sufficient that there exist a function $V(t, \mathbf{x})$ satisfying the conditions:

(a) $V(t, \mathbf{x})$ is defined in the region

$$\sum_{i=1}^{m} x_i^2 + \sum_{j=m+1}^{n} (x_j / A_j)^2 \leqslant r^2 > 0, \qquad t \geqslant 0$$

(b) for any $C_1 > 0$ we can find $C_2 > 0$ such that from

$$\sum_{i=1}^{m} x_i^2 > C_1^2, \qquad \sum_{j=m+1}^{n} (x_j / A_j)^2 > C_1^2$$

follows $V(t, \mathbf{x}) > C_2$ $(C_1^2 < r^2 / 2)$;

(c) for any $\gamma_1 > 0$ we can find $\gamma_2 > 0$ such that $V(t, \mathbf{x}) < \gamma_1$ if only

$$\sum_{i=1}^{m} x_i^2 < \gamma_2^2, \qquad \sum_{j=m+1}^{n} (x_j / A_j)^2 < \gamma_2^2$$

(d) the function V does not increase along the solutions of system (1.1), on which it is still defined.

P r o o f. S u f f i c i e n c y. Under the change of arguments of the function $V(t, \mathbf{x})$ by formulas (2.8) we obtain the function $V(t, \xi)$ satisfying the conditions of the theorem given in [4] (p. 29), therefore the motion $\xi = 0$ of system (2.9) is stable uniformly in t_0. By virtue of Lemma 4 the motion $\mathbf{x} = 0$ of system (1.1) is $\mathbf{y}$-stable uniformly in t_0.

N e c e s s i t y. From the uniform in t_0 $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$ of system (1.1) it follows, according to Lemma 4, that the motion $\xi = 0$ of system (2.9) is stable uniformly in t_0. Having taken for system (2.9) a function $V(t, \xi)$ in accordance with the theorem in [4] (p. 29) and having replaced its arguments by formulas (2.8), we obtain the required function $V(t, \mathbf{x})$. The theorem is proved.

C o r o l l a r y [4]. If the estimates (2.7) are not known beforehand, but there exists a function V satisfying the hypotheses of Theorem 5 (in which now the $A_i(t)$ are some functions continuously differentiable and positive for $t \geqslant 0$) then the motion $\mathbf{x} = 0$ of system (1.1) is $\mathbf{y}$-stable uniformly in t_0 and the estimates (2.7) hold.

4. Let $V(t, \mathbf{x})$ satisfy the differential inequality [6, 22]

$$V^{\bullet}(t, \mathbf{x}) \leqslant \omega(t, V(t, \mathbf{x})) \tag{2.10}$$

in which $\omega(t, v)$ is a function continuous for $t \geqslant 0$, $v \geqslant 0$ and the congruence equation

$$v^{\cdot} = \omega(t, v) \qquad (\omega(t, 0) \equiv 0) \tag{2.11}$$

has the unique solution $v = v(t; t_0, v_0)$ satisfying the initial condition $v(t_0; t_0, v_0) = v_0$ for each point (t_0, v_0) of the domain.

T h e o r e m 6. [6]. If a function $V(t, \mathbf{x})$ exists satisfying inequalities (1.7) and (2.10) and, furthermore,

(1) the solution $v = 0$ of Eq. (2.11) is stable, then the motion $\mathbf{x} = 0$ of system (1.1) is $\mathbf{y}$-stable;

(2) V satisfies inequality (1.10) and solution $v = 0$ of Eq. (2.11) is stable uniformly in t_0, then the motion $\mathbf{x} = 0$ is $\mathbf{y}$-stable uniformly in t_0.

P r o o f. From (2.10) it follows [23] that when $V(t_0, \mathbf{x}_0) \leqslant v_0$,

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant v(t; t_0, v_0), \qquad t \geqslant t_0 \tag{2.12}$$

(1) For any $\varepsilon > 0$, $t_0 \geqslant 0$ there exists, by virtue of the stability of the solution $v = 0$, $\eta(\varepsilon, t_0) > 0$ such that from $v_0 < \eta$ follows $v(t; t_0, v_0) < a(\varepsilon)$ for all $t \geqslant t_0$. Let

$\delta(\eta, t_0) = \delta(\varepsilon, t_0) > 0$ be such that $V(t_0, x_0) < \eta$ if $\|x_0\| < \delta$. Then from (2.12) and (1.7) follows, for $t \geqslant t_0$, $\|x_0\| < \delta$

$$a(\|y(t; t_0, x_0)\|) \leqslant V(t, x(t; t_0, x_0)) \leqslant v(t; t_0, v_0) < a(\varepsilon)$$

whence $\|y(t; t_0, x_0)\| < \varepsilon$.

(2) In this case $\eta(\varepsilon) > 0$ does not depend on t_0; but then also $\delta(\varepsilon) = b^{-1}(\eta(\varepsilon))$ does not depend on t_0. The theorem is proved.

The converse assertion is valid for the first part of Theorem 6. Together with system (1.1) we consider the system [12]

$$x^{*\bullet} = X(t, x^*)\varphi(y^*) \equiv X^*(t, x^*) \tag{2.13}$$

in which φ is a scalar continuously-differentiable function, $0 \leqslant \varphi \leqslant 1$ and

$$\varphi(y^*) = \begin{cases} 1 & \text{for} \quad \|y^*\| \leqslant h \\ 0 & \text{for} \quad h < H_1 \leqslant \|y^*\| \leqslant H \end{cases}$$

Let $x^* = x^*(t; t_0, x_0^*)$ be the solution of system (2.13) with the initial conditions $x^*(t_0; t_0, x_0^*) = x_0^*$. We assume that solution of system (2.13) is z^*-extendible.

By $V_{(1)}$ and $V_{(2)}$ we denote the derivatives of function V by virtue of systems (1.1) and (2.13) respectively. These systems coincide in the region

$$t \geqslant 0, \quad \|y\| \leqslant h, \quad 0 \leqslant \|z\| < \infty \tag{2.14}$$

In the region (2.14) we seek the function $V(t, x)$ among the solutions of the functional equation [6]

$$V_{(1)}^{\bullet}(t, x) \equiv V_{(2)}^{\bullet}(t, x) = \omega(t, V(t, x)) \tag{2.15}$$

We assume that the continuous derivatives $\partial\omega / \partial v$, $\partial X_i / \partial x_j$ $(i, j = 1, \ldots, n)$ exist. The solution of Eq. (2.15), satisfying the condition $V(0, x) = \mu(x)$ (μ is a differentiable function), is given by the formula [6]

$$V(t, x) = v(t; 0, \mu(x^*(0; t, x))) \tag{2.16}$$

By virtue of the differentiability of the solutions of system (2.13) and of Eq. (2.11) with respect to the initial conditions, V has the continuous derivatives $\partial V / \partial t$, $\partial V / \partial x_i$ $(i = 1, \ldots, n)$ (*). Let the function ω satisfy the condition —

A) all solutions of Eq. (2.11) are defined for $t \in [0, \infty)$ and the function $v(t; 0, v_0)$ is positive-definite,

$$v(t; 0, v_0) \geqslant \lambda(v_0) \tag{2.17}$$

T h e o r e m 7 [6]. If the motion $x = 0$ of system (1.1) is y-stable, then for any function ω satisfying condition (A) there exists a function $V(t, x)$ satisfying inequality (1.7) and Eq. (2.15).

P r o o f. It is sufficient to prove that under a suitable choice of the function $\mu(x)$ the function V defined by formula (2.16) will satisfy inequality (1.7). Let $\mu(x)$ be such that

$$\mu(x) \geqslant \mu^*(\|x\|) \tag{2.18}$$

Using method given in [24, 25] we can show [12] that

$$x^*(0; t, x)\| \geqslant \nu(\|y\|) \quad (^{**}) \tag{2.19}$$

---

*) If we waive the smoothness of the functions mentioned, then V proves to be not differentiable, but the derivative $dV(t, x(t; t_0, x_0)) / dt$ will exist [6].

**) $\lambda(r)$, $\mu^*(r)$ and $\nu(r)$ are functions of the type of $a(r)$.

follows from the condition that the motion $x = 0$ is y-stable. From (2.17) – (2.19) ensues (1.7) with $a(r) = \lambda(\mu^*(\nu(r)))$. The theorem is proved.

5. Let us consider the application of a vector-valued function V. The results obtained in [26] for $m = n$ were carried over (without proof) to the case $m < n$ in [9]. Let ω, ψ, f be vectors in the k-dimensional space R^k. We write $\omega \leqslant \psi$ if $\omega_i \leqslant \psi_i$ $(i = 1, \ldots, \ldots, k)$. By definition the function $f_s(t, \omega)$ does not decrease with respect to $\omega_1, \ldots, \omega_{s-1}, \omega_{s+1}, \ldots \omega_k$ if $f_s(t, \omega^*) \leqslant f_s(t, \omega^{**})$ for $\omega_s^* = \omega_s^{**}$, $\omega_i^* \leqslant \omega_i^{**}$ $(i = 1, \ldots, s-1, s+1, \ldots, k)$.

We introduce the following conditions:

I. There exists a vector-valued function $V = (V_1, \ldots, V_k)$ such that

1) V and V˙ are continuous, $V(t, 0) \equiv V^{\cdot}(t, 0) \equiv 0$;

2) $V_1 \geqslant 0, \ldots, V_l \geqslant 0$ for certain* l, $1 \leqslant l \leqslant k$, while

$$V_1(t, x) + \ldots + V_l(t, x) \geqslant a(\| y \|) \tag{2.20}$$

3) the derivative V˙ satisfies the inequality

$$V^{\cdot}(t, x) \leqslant f(t, V(t, x))$$

II. 1) The vector-valued function $f(t, V)$ is defined and is continuous in the region

$$t \geqslant 0, \qquad \| V \| < R$$

where $R = \infty$ or $R > \sup \{\| V(t, x) \| : t \geqslant 0, \| y \| \leqslant H\}$;

2) Each of the functions f_s $(s = 1, \ldots, k)$ does not decrease with respect to $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$;

3) $f(t, 0) \equiv 0$.

III. Let $\alpha = (\omega_1, \ldots, \omega_l)$. Consider the congruence system

$$\omega^{\cdot} = f(t, \omega) \tag{2.21}$$

Under the conditions $\omega_{10} \geqslant 0, \ldots, \omega_{l0} \geqslant 0$ the solution $\omega = 0$ of system (2.21) is:

1) α-stable;

2) α-stable uniformly in t_0;

3) asymptotically α-stable;

4) asymptotically α-stable uniformly in $\{t_0, \omega_0\}$.

IV. $V(t, x) \to 0$ uniformly in $t \geqslant 0$ as $x \to 0$.

Theorem 8 [9]. (1) If conditions I, II and III-1 are fulfilled, then the motion $x = 0$ is y-stable.

(2) If conditions I, II, III-2 and IV are fulfilled, then the motion $x = 0$ is y-stable uniformly in t_0.

Proof [26]. (1) For any $\varepsilon > 0$, $t_0 \geqslant 0$ we can find $\lambda(\varepsilon, t_0) > 0$ such that from

$$\sum_{s=1}^{k} | \omega_{s0} | < \lambda$$

(for $\omega_{10} \geqslant 0, \ldots, \omega_{l0} \geqslant 0$) follows:

$$\sum_{s=1}^{l} | \omega_s(t; t_0, \omega_0) | < a(\varepsilon) \quad (*)$$

*) $\omega = \omega(t; t_0, \omega_0)$ is the solution of system (2.21), satisfying the initial conditions $\omega(t_0; t_0, \omega_0) = \omega_0$.

for all $t \geqslant t_0$. By virtue of II-2 the theorem of [23] is applicable, according to which there exists the upper solution $\omega^+ (t; t_0, \omega_0)$ with the same initial conditions, satisfying for $t \geqslant t_0$ the inequality

$$\sum_{s=1}^{l} | \omega_s^+ (t; t_0, \omega_0) | < a(\varepsilon)$$

With respect to λ and t_0 we can select $\delta(\lambda, t_0) = \delta(\varepsilon, t_0) > 0$ such that from $\| x_0 \| < \delta$ follows:

$$\sum_{j=1}^{k} | V_j (t_0, x_0) | < \lambda$$

Let us show that $\| y (t; t_0, x_0) \| < \varepsilon$ for $t \geqslant t_0$, if $\| x_0 \| < \delta$. We assume that there exist $t_* > t_0$ and x_* with $\| x_* \| < \delta$ for which $\| y (t; t_0, x_*) \| < \varepsilon$ for $t \in [t_0, t_*)$, but

$$\| y (t_*; t_0, x_*) \| = \varepsilon \tag{2.22}$$

We set $\omega_* = V (t_0, x_*)$ (here $\omega_{1*} \geqslant 0, \ldots, \omega_{l*} \geqslant 0$). By the choice of δ

$$\sum_{s=1}^{k} | \omega_{s*} | < \lambda$$

consequently, for $t \in [t_0, t_*] \subset [t_0, \infty)$

$$\sum_{s=1}^{l} | \omega_s^+ (t; t_0, \omega_*) | < a(\varepsilon)$$

The functions $V_s (t, x (t; t_0, x_*))$ are continuously differentiable in t on $[t_0, t_* + \Delta t)$ ($\Delta t > 0$ is sufficiently small). By virtue of I-3,

$$V^{\bullet} (t, x (t; t_0, x_*)) \leqslant f (t, V (t, x (t; t_0, x_*)))$$

then, according to the theorem of [23],

$$V_s (t, x (t; t_0, x_*)) \leqslant \omega_s^+ (t; t_0, \omega_*)$$

Hence

$$a (\| y (t; t_0, x_*) \|) \leqslant \sum_{s=1}^{l} V_s (t, x (t; t_0, x_*)) \leqslant \sum_{s=1}^{l} \omega_s^+ (t; t_0, \omega_*) < a(\varepsilon) \tag{2.23}$$

Consequently, $\| y (t; t_0, x_0) \| < \varepsilon$ for all $t \in [t_0, t_*]$ which contradicts equality (2.22) when $t = t_*$.

(2) In this case λ and δ may be chosen independent of t_0. The theorem is proved.

6. In the problem of the preservation of stability relative to a part of the variables [6], together with system (1.1) we can consider the system

$$x = X (t, x) + R (t, x) \qquad (R (t, 0) \equiv 0) \tag{2.24}$$

satisfying the same conditions as does system (1.1). Let $V_{(1)}$ and $V_{(2)}$ be the derivatives of function V by virtue of (1.1) and (2.24).

Theorem 9 [6]. If a function $V (t, x)$ satisfies inequality (1.7) and the Lipschitz condition

$$| V (t, x_1) - V (t, x_2) | \leqslant L \| x_1 - x_2 \| \tag{2.25}$$

while $V_{(1)} \leqslant 0$, then we can find a function $d(r)$ (of the type of $a(r)$) such that from the inequalities

$$\| R (t, x) \| \leqslant \varphi (t) d (\| y \|) \tag{2.26}$$

$$\int_0^{\infty} \varphi (t) dt < +\infty \tag{2.27}$$

follows the uniform in t_0 y-stability of the motion $x = 0$ of system (2.24).

Proof. By virtue of (2) of Theorem 1 the motion $x = 0$ of system (1.1) is y-stable uniformly in t_0 and we can talk about the preservation of stability. From (2.25) and (2.26) follows:

$$V_{(2)}^{\cdot}(t, x) \leqslant V_{(1)}^{\cdot}(t, x) + L\varphi(t)\, d(\|y\|) \leqslant L\varphi(t)\, d(\|y\|) \tag{2.28}$$

whence, by virtue of (1.7), $V_{(2)}^{\cdot}(t, x) \leqslant L\, \varphi(t)\, d(a^{-1}(V(t, x)))$. If the function $d(a^{-1}(r)) = \rho(r)$ is such that

$$\int_0 \frac{dr}{\rho(r)} = +\infty$$

then the solution $v = 0$ of the equation $v^{\cdot} = L\, \varphi(t)\, \rho(v)$ is stable uniformly in t_0. Hence follows the required result if we set, for example, $d(r) = a(r)$.

7. In order to detect the y-instability of the motion $x = 0$ of system (1.1) it is sufficient to observe only one trajectory emerging onto the surface $\|y\| = H$ for arbitrarily small $\|x_0\|$ [21]. The set of points (t, x) of region (1.2) for which $V(t, x) > 0$ is called the region $V > 0$ [21].

Definition 6 [21]. A function $U(t, x)$ is said to be positive-definite in the region $V > 0$ if for arbitrarily small $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for every point (t, x) of region (1.2), satisfying the condition $V(t, x) \geqslant \varepsilon$, the inequality $U(t, x) \geqslant \delta$ is fulfilled.

As was noted in [3, 14], Chetaev's theorem [21] on instability can be successfully applied to the y-instability problem:

Theorem 10 [21]. If a function $V(t, x)$ is bounded in the region $V > 0$, existing for any $t \geqslant 0$ and for arbitrarily small $\|x\|$, and if $V^{\cdot}$ is a positive-definite function in the region $V > 0$, then the motion $x = 0$ is y-unstable.

The statement and the proof of this theorem coincide with those of Chetaev's theorem [21] with the sole difference that the region considered in [21] was $t \geqslant 0$, $\|x\| \leqslant H > 0$, while in Theorem 10 (and in Definition 6) we consider the region (1.2). The conditions of Theorem 10 ensure that the corresponding solutions will leave region (1.2) in a time not exceeding $(L - v_0)/l'$ [21], starting from the instant t_0. Since this time is finite, the solutions go onto the surface $\|y\| = H$ (see condition b of Sect. 1).

Note. Theorem 10 remains valid [14] if the function satisfying the hypotheses of Theorem 10 is $V = V(t, y)$.

Theorem 11 [13]. If: (1) system (1.1) is autonomous and all its solutions starting in some neighborhood of the point $x = 0$ are z-bounded;

(2) the function $V(x)$ is such that: $V(0) = 0$ and in any neighborhood of the origin there exists a point x for which $V(x) < 0$;

$$(3)\quad V^{\cdot}(x) = 0 \text{ for } x \in M, \quad V^{\cdot}(x) < 0 \text{ for } x \bar{\in} M \tag{2.29}$$

where M is the set not containing entire trajectories besides $x = 0$, then the motion $x = 0$ is y-unstable.

Proof. Assume the contrary: let the motion $x = 0$ be y-stable. Having chosen x_0 from the conditions $V(x_0) < 0$, $\|y(t; 0, x_0)\| < H$ for $t \geqslant 0$, we obtain, by virtue of (1) and (2),

$$\|x(t; 0, x_0)\| \geqslant \eta > 0, \qquad t \geqslant 0 \tag{2.30}$$

The set Γ^+ of the ω-limit points of the solution $x(t; 0, x_0)$ is nonempty and invariant [27], moreover, $\Gamma^+ \subset M$ [28, 29]. By virtue of (2.30), Γ^+ does not contain the point $x = 0$. Consequently, the set M contains a trajectory other than $x = 0$, which is impossible. The theorem is proved.

This result generalizes a theorem of Krasovskii [20].

Theorem 12 [13]. If conditions 1 and 2 of Theorem 11, (2.29) and

4) $V(0, z) \geqslant 0$ for any z;

5) the set $\{x: y = 0\}$ is invariant;

6) the set $M \setminus \{x: y = 0\}$ does not contain entire trajectories, then the motion $x = 0$ is y-unstable.

Proof. We assume the contrary and we choose x_0 as in the proof of Theorem 11. The set Γ^+ is not empty. Let

$$\lim_{n\to\infty} x(t_n; 0, x_0) = x_* \in \Gamma^+.$$

If

$$\lim_{t\to\infty} \| y(t; 0, x_0) \| = 0$$

then $y_* = 0$ and, by passing to the limit in the inequalities

$$\lim_{t\to\infty} V(x(t; 0, x_0)) \leqslant V(x_0) < 0$$

we obtain $0 \leqslant V(0, z_*) \leqslant V(x_0) < 0$, which is impossible. Consequently, $\| y(t_n; 0, x_0) \| \geqslant \eta > 0$ for some sequence $t_n \to \infty$ and we can take $y_* \neq 0$. According to 5, $\| y(t; 0, x_*) \| \neq 0$ for all $t \geqslant 0$, whence, by virtue of the invariance of Γ^+ and of the property $\Gamma^+ \subset M$ follows $x(t; 0, x_*) \in M \setminus \{x: y = 0\}$ for any $t \geqslant 0$, which is impossible. The theorem is proved.

3. Asymptotic stability. 1. Theorem 13 [10, 14]. If a function $V(t, x)$ satisfying inequality (1.7) exists and if for any $t_0 \geqslant 0$ we can find $\Delta(t_0) > 0$ such that from $\| x_0 \| < \Delta$ follows $V(t, x(t; t_0, x_0)) \downarrow 0$(*) as $t \to \infty$, then the motion $x = 0$ is asymptotically y-stable.

This assertion follows from the y-stability, the inequality (1.7) and the condition $V \downarrow 0$.

Theorem 14 [10, 14]. If a function $V(t, x)$ is such that

$$V(t, x) \geqslant \theta(t)\, a(\| y \|)$$

for a function $\theta(t)$ monotonically increasing to infinity, $\theta(0) = 1$, while $V^{\cdot} \leqslant 0$, then the motion $x = 0$ is asymptotically y-stable.

This result generalizes a theorem of Chetaev [21]. For each pair of numbers $t_0 \geqslant 0$, $\delta' > 0$ we consider the set [9]

$$E(t_0, \delta') = \{(t, x) : t \geqslant t_0,\ x = x(t; t_0, x_0),\ \| x_0 \| < \delta'\}$$

Theorem 15 [9]. If a function $V(t, x)$ satisfying the hypotheses of Theorem 2 exists and if for any $t_0 \geqslant 0$ we can find $\delta(t_0) > 0$ and $M(t_0) > 0$ such that

$$\| Y(t, x) \| \leqslant M \quad \text{for} \quad (t, x) \in E(t_0, \delta') \tag{3.1}$$

then the motion $x = 0$ is asymptotically y-stable.

Proof. By virtue of Theorem 1 (1), for every $\varepsilon > 0$, $t_0 \geqslant 0$ we can find $\delta(\varepsilon, t_0)$, $0 < \delta < \delta'$, such that from $\| x_0 \| < \delta$ follows $\| y(t; t_0, x_0) \| < \varepsilon$ for $t \geqslant t_0$. Let us show that $\lim \| y(t; t_0, x_0) \| = 0$ as $t \to \infty$ if $\| x_0 \| < \delta$. Assume the contrary: let there exist a point x_* with $\| x_* \| < \delta$, a number $l > 0$ and a sequence $t_k \to \infty$, $t_k - t_{k-1} \geqslant \alpha > 0$, $k = 1, 2, 3, \ldots$, such that $\| y(t_k; t_0, x_*) \| \geqslant l$. By virtue of (3.1) we can choose

*) The notation "$V \downarrow 0$" means that "V tends to zero, decreasing monotonically (in the wide sense)".

β, $0 < \beta < \alpha / 2$, for which $l / 2 \leqslant \| \mathbf{y}(t;\ t_0,\ \mathbf{x}_*) \| < \varepsilon$ for $t \in [t_k - \beta,\ t_k + \beta]$, $k = 1, 2, 3, \ldots$. Here, on the basis of (2.5), from (2.2) follows

$$0 \leqslant V(t_k + \beta,\ \mathbf{x}(t_k + \beta;\ t_0,\ \mathbf{x}_*)) \leqslant V(t_0,\ \mathbf{x}_*) + \sum_{i=1}^{k} \int_{t_i - \beta}^{t_i + \beta} V^{\cdot}(\tau,\ \mathbf{x}(\tau;\ t_0,\ \mathbf{x}_*))\, d\tau \leqslant$$
$$\leqslant V(t_0,\ \mathbf{x}_*) - 2k\beta c\,(l / 2)$$

which is impossible for k sufficiently large. The theorem is proved.

Theorem 15 generalizes a theorem given in [30]. By an example it is shown in [12] that when $m < n$ Theorem 15 is not invertible even in the case of asymptotic $\mathbf{y}$-stability uniform in $\{t_0,\ \mathbf{x}_0\}$. It can be shown [31] that theorem given in [30] (i.e., Theorem 15 with $m = n$) is also not invertible in the general case.

2. Theorem 16 [32]. If a function $V(t, \mathbf{x})$ satisfies inequalities (1.7) and (1.9), i.e.,

$$a(\| \mathbf{y} \|) \leqslant V(t,\ \mathbf{x}) \leqslant b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right), \qquad m \leqslant k \leqslant n \tag{3.2}$$

and its derivative

$$V^{\cdot}(t,\ \mathbf{x}) \leqslant -c\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right) \tag{3.3}$$

then the motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable.

Proof. For every $\varepsilon > 0$, $t_0 \geqslant 0$ we select $\delta(\varepsilon,\ t_0) > 0$ in accordance with Theorem 1 (1). We show that from $\| \mathbf{x}_0 \| < \delta$ follows $\lim V(t, \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) = 0$ as $t \to \infty$. If we assume the contrary, then by virtue of $V^{\cdot} \leqslant 0$ we have $V(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) \geqslant V_* > 0$. On the basis of this, from (3.2) and (3.3) we conclude that

$$\left(\sum_{i=1}^{k} x_i^2(t;\ t_0,\ \mathbf{x}_0)\right)^{1/2} \geqslant b^{-1}(V_*), \qquad V^{\cdot}(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) \leqslant -c\,(b^{-1}(V_*)) \tag{3.4}$$

Using (3.4) and (2.2) we obtain

$$0 \leqslant V(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) \leqslant V(t_0,\ \mathbf{x}_0) - c\,(b^{-1}(V_*))\,(t - t_0)$$

which is impossible for t sufficiently large. Thus, $\lim V(t,\ \mathbf{x}(t; t_0,\ \mathbf{x}_0)) = 0$ as $t \to \infty$. The result required follows from Theorem 13.

Definition 7 (cf. [12]). The solutions of system (1.1) possess property (R) if for some $\delta > 0$ and for any $\varepsilon > 0$ we can find $T(\varepsilon) > 0$ such that from

$$t_0 \geqslant 0,\ \sum_{i=1}^{k} x_{i0}^2 < \delta^2 \ (-\infty < x_{j0} < +\infty,\ j = k+1, \ldots, n)$$

follows $\| \mathbf{y}(t;\ t_0,\ \mathbf{x}_0) \| < \varepsilon$ for all $t \geqslant t_0 + T$.

Analogous to [12] we can prove

Theorem 17. For the existence of a function V satisfying the hypotheses of Theorem 16 it is necessary that the solutions of system (1.1) possess property (R) and that the identities

$$X_i(t, 0, \ldots, \quad 0, x_{k+1}, \ldots, x_n) \equiv 0 \qquad (i = 1, \ldots, k)$$

be fulfilled.

If $k = m$, while X_i and $\partial X_i / \partial x_j$ $(i,\ j = 1, \ldots, n)$ are continuous and bounded in region (1.2), then these conditions are sufficient [12] for the existence of a function V satisfying the hypotheses of Theorem 16. Theorem 16 is noninvertible when $k > m$ which follows from the example in [12].

3. Theorem 18 [9]. If a function $V(t, x)$ satisfying inequalities (1.7) and (1.10) exists, i.e.,

$$a(\|y\|) \leqslant V(t, x) \leqslant b(\|x\|) \tag{3.5}$$

the derivative $V^{\cdot} \leqslant 0$ and $V(t, x) \to 0$ uniformly as $V^{\cdot}(t, x) \to 0$ (*), then the motion $x = 0$ is asymptotically y-stable uniformly in $\{t_0, x_0\}$.

Proof. If $t_0 \geqslant 0$, $\|x_0\| < \Delta_0 = b^{-1}(a(H))$, then the inequality $\|y(t; t_0, x_0)\| < H$ is valid for any $t \geqslant t_0$. For every ε, $0 < \varepsilon < H$ we can find $\delta'(\varepsilon) > 0$ such that $V(t, x) < a(\varepsilon)$ follows from $|V^{\cdot}(t, x)| < \delta'$. We set $T(\varepsilon) = 2a(\varepsilon)/\delta'(\varepsilon)$. If we admit that $|V^{\cdot}(\tau, x(\tau; t_0, x_0))| \geqslant \delta'(\varepsilon)$ for $\tau \in (t_0, t_0 + T)$ and $\|x_0\| < \Delta_0$, then from (2.2) we obtain

$$0 \leqslant V(t_0 + T, x(t_0 + T; t_0, x_0)) \leqslant V(t_0, x_0) - 2a(\varepsilon) \leqslant a(\varepsilon) - 2a(\varepsilon) < 0$$

which is impossible. Thus, for some $t_* \in (t_0, t_0 + T)$ we have $|V^{\cdot}(t_*, x(t_*; t_0, x_0))| < \delta'$ and, consequently, $V(t_*, x(t_*; t_0, x_0)) < a(\varepsilon)$. But then for $t \geqslant t_*$

$$a(\|y(t; t_0, x_0)\|) \leqslant V(t, x(t; t_0, x_0)) \leqslant V(t_*, x(t_*; t_0, x_0)) < a(\varepsilon)$$

whence $\|y(t; t_0, x_0)\| < \varepsilon$ for all $t \geqslant t_0 + T > t_*$. The theorem is proved.

4. We consider some generalizations (see Sect. 2, subsection 2).

Theorem 19 [5]. If there exists a function $V(t, x)$ possessing the properties:

a) $V(t, 0) \equiv 0$, $V(t, x)$ is continuous at the point $x = 0$;

b) V satisfies inequality (1.7);

c) $$D^+V(t, x(t; t_0, x_0)) \leqslant -c(V(t, x(t; t_0 x_0))) \quad (^{**}) \tag{3.6}$$

then the motion $x = 0$ is asymptotically y-stable.

Proof. Having chosen $\delta(\varepsilon, t_0) > 0$, in accordance with Theorem 1 (1) we get that the limit $\lim V(t, x(t; t_0, x_0)) = V_* \geqslant 0$ as $t \to \infty$ exists for $\|x_0\| < \delta$. If we assume that $V_* > 0$, then

$$D^+V(t, x(t; t_0, x_0)) \leqslant -c(V_*)$$

follows from (3.6). Integrating this relation we find

$$0 \leqslant V(t, x(t; t_0, x_0)) \leqslant V(t_0, x_0) - c(V_*)(t - t_0)$$

which is impossible for t sufficiently large. Thus, $V_* = 0$, which is what we had to prove.

Theorem 20 [5]. If a function $V(t, x)$ satisfying conditions a), b) of Theorem 19 and inequality (1.10) exists and, moreover,

$$D^+V(t, x(t; t_0, x_0)) \leqslant -c(\|x(t; t_0, x_0)\|) \tag{3.7}$$

then the motion $x = 0$ is asymptotically y-stable uniformly in $\{t_0, x_0\}$.

Proof. Inequalities (3.5) hold by hypothesis. Let $\varepsilon > 0$ be given. If $\delta(\varepsilon) = b^{-1}(a(\varepsilon))$, then for $\|x_0\| < \delta$

$$a(\|y(t; t_0, x_0)\|) \leqslant V(t, x(t; t_0, x_0)) \leqslant V(t_0, x_0) \leqslant b(\|x_0\|) < a(\varepsilon)$$

---

*) This means that for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $|V^{\cdot}(t, x)| < \delta$ follows from $V(t, x) < \varepsilon$.

**) The quantity $D^+V(t, x(t; t_0, x_0)) = \overline{\lim_{h\to+0}}\, [V(t + h, x(t + h; t_0, x_0)) - V(t, x(t; t_0, x_0))]/h$ is Dini's right upper-derived number [33, 34] of the function $V(t, x(t; t_0, x_0))$.

whence $\| y(t;\ t_0, x_0) \| < \varepsilon$ for $t \geqslant t_0$. We set $\Delta_0 = \delta(H)$, $T(\varepsilon) = b(\Delta_0)/c(\delta(\varepsilon))$ and we let $\| x_0 \| < \Delta_0$, $t_0 \geqslant 0$. If we assume that $\| x(t;\ t_0, x_0) \| \geqslant \delta(\varepsilon)$ for all $t \in (t_0, t_0 + T)$, then, by integrating (3.7) we obtain

$$0 \leqslant V(t_0 + T, x(t_0 + T;\ t_0, x_0)) \leqslant V(t_0, x_0) - c(\delta(\varepsilon))\,T < b(\Delta_0) - c(\delta(\varepsilon))\,T = 0$$

which is impossible. Consequently, there exists an instant $t_* \in (t_0,\ t_0 + T)$ for which $\| x(t_*;\ t_0, x_0) \| < \delta(\varepsilon)$. But in such a case $\| y(t;\ t_0, x_0) \| < \varepsilon$ for all $t \geqslant t_0 + T > t_*$, which is what we had to prove.

5. Theorem 21 [4]. In order for the motion $x = 0$ to be y-stable uniformly in t_0 and asymptotically y-stable, it is necessary and sufficient that there exist a function $V(t, x)$ satisfying the hypotheses of Theorem 5 and $V(t, x(t;\ t_0, x_0)) \downarrow 0$.

Proof. 1) Sufficiency. The hypotheses of Theorem 5 are fulfilled. From $V \downarrow 0$ it follows that $\| y(t;\ t_0, x_0) \| \to 0$ as $t \to \infty$ if $\| x_0 \|$ is sufficiently small.

2) Necessity. From the hypotheses it follows that there exists a function $V(t, x)$ satisfying the hypotheses of Theorem 5. By the construction of the function V ([4], p.29) and by virtue of the asymptotic y-stability, $V(t, x(t;\ t_0, x_0)) \downarrow 0$.

6. We consider the application of the differential inequalities (see Sect. 2, subsection 4).

Theorem 22 [6]. If a function $V(t, x)$ exists satisfying inequalities (2.10) and (1.7) and, furthermore, if:

1) the solution $v = 0$ of Eq. (2.11) is asymptotically stable, then the motion $x = 0$ of system (1.1) is asymptotically y-stable;

2) V satisfies inequality (1.10) and the solution $v = 0$ of Eq. (2.11) is asymptotically stable uniformly in $\{t_0, v_0\}$, then the motion $x = 0$ of system (1.1) is asymptotically y-stable uniformly in $\{t_0,\ x_0\}$.

Proof. 1) From (2.12) and (1.7) it follows that $a(\| y(t;\ t_0, x_0) \|) \leqslant v(t;\ t_0, v_0)$. Since $\lim v(t;\ t_0, v_0) = 0$ as $t \to \infty$, also $\lim \| y(t;\ t_0, x_0) \| = 0$ if $\| x_0 \|$ is sufficiently small.

2) Let $\eta_0 > 0$ and $T(a(\varepsilon)) = T(\varepsilon)$ be the number appearing in the definition of the uniform asymptotic stability of the solution $v = 0$ of Eq. (2.11). We choose $\Delta_0 > 0$ from the condition $b(\Delta_0) < \eta_0$. Then, for $\| x_0 \| < \Delta_0$, $t \geqslant t_0 + T$ we have

$$a(\| y(t;\ t_0, x_0) \|) \leqslant V(t, x(t;\ t_0, x_0)) \leqslant v(t;\ t_0, v_0) < a(\varepsilon)$$

whence $\| y(t;\ t_0, x_0) \| < \varepsilon$. The theorem is proved.

Let us consider a vector-valued function V (see Sect. 2, subsection 2).

Theorem 23 [9]. 1) If conditions I, II and III-3) are fulfilled, then the motion $x = 0$ is asymptotically y-stable.

2) If conditions I, II, III-4) and IV are fulfilled, then the motion $x = 0$ is asymptotically y-stable uniformly in $\{t_0,\ x_0\}$.

Proof. 1) Since $\lim_{t \to \infty} \sum_{s=1}^{l} \omega_s^+(t, t_0, \omega_0) = 0$, then (1.3) follows from (2.23) if $\| x_0 \|$ is sufficiently small.

2) In this case relation (1.3) is fulfilled uniformly in $\{t_0, x_0\}$. The theorem is proved.

7. In this subsection $\dot{V}_{(1)}$ and $\dot{V}_{(2)}$ denote the derivatives of the function V relative to systems (1.1) and (2.24) (see Sect. 2, subsection 6).

Theorem 24 [6]. If there exists a bounded function $V(t, x)$ satisfying inequality (1.7) and the Lipschitz condition (2.25) and, moreover, if $\dot{V}_{(1)}(t, x) \leqslant -c(\| x \|)$, then we can find a function $d(r)$ such that from the conditions (2.26) and (3.8),

$$\left\{-t+\alpha\int_{t_0}^{t_0+t}\varphi(\tau)\,d\tau\right\}\underset{t_0\geqslant 0}{\rightrightarrows}-\infty \quad \text{as} \quad t\to\infty \quad (\alpha>0) \tag{3.8}$$

follows the uniform in $\{t_0, x_0\}$ asymptotic y-stability of motion $x = 0$ of system (2.24).

P r o o f. Under the conditions imposed on V we can construct [22] a function $W(t, x)$ such that

$$W(t,0)\equiv 0, \quad p(\|y\|)\leqslant W(t,x)\leqslant\|x\|$$
$$W_{(1)}(t,x)\leqslant -W(t,x) \tag{3.9}$$

$p(r)$ is a function of the type of $a(r)$; furthermore, W satisfies a Lipschitz condition in x with a Lipschitz constant equal to unity.
Hence, by virtue of (2.26) and (3.9) follows

$$W_{(2)}(t,x)\leqslant -W(t,x)+\varphi(t)\,d\,(\|y\|)$$

Let $d(r)\leqslant Cp(r)$, $0<C\leqslant\alpha$. Then

$$W_{(2)}(t,x)\leqslant[-1+\alpha\varphi(t)]\,W(t,x) \tag{3.10}$$

On the basis of (3.8) and of item 2) of Theorem 22, from (3.10) we conclude that the motion $x = 0$ of system (2.24) is asymptotically y-stable uniformly in $\{t_0, x_0\}$. The theorem is proved.

8. Let us consider criteria based on functions having a sign-constant derivative [20, 29, 35, 28]. In this subsection we assume that system (1.1) is autonomous and, consequently, its solutions possess the group property

$$x(t;t_0,x_0)\equiv x(t+\tau;t_0+\tau,x_0) \tag{3.11}$$

and, moreover, that all its solutions starting in some neighborhood of the point $x = 0$ are bounded.

T h e o r e m 2 5 [10, 13, 14]. If a function $V(x)$ is such that $V(x)\geqslant a(\|y\|)$, while $V^{\cdot}$ satisfies condition (2.29) where M is the set of points $\{x\}$ not containing entire trajectories, and furthermore if $x = 0$, then the motion $x = 0$ is asymptotically y-stable.

P r o o f [10, 14]. Having been given a number ε, $0<\varepsilon<H$, we choose $\delta(\varepsilon)>0$ in accordance with item 2) of Theorem 1. Let us show that from $\|x_0\|<\delta$ it follows that $\lim V(x(t;\ t_0,\ x_0)) = 0$ as $t\to\infty$. Assuming that this is false, by virtue of $V^{\cdot}\leqslant 0$ we obtain

$$V(x(t;t_0,x_0))\geqslant V_*>0 \tag{3.12}$$

The solution $x(t;\ t_0,\ x_0)$, being bounded by hypothesis, has a limit point x_*: $x(t_0+k\tau;\ t_0, x_0)\to x_*$, $k=k_1, k_2, k_3,\dots, k_i\to\infty$, $\tau = \text{const}>0$; by the continuity of $V(x_*) = V_*$. Since the solution $x(t;\ t_0,\ x_*)$ does not lie wholly in the set M, for some $T>t_0$ we have $V(x(T;\ t_0,\ x_*))<V_*$. Since $x(t_0+k\tau;\ t_0,\ x_0)\to x_*$, by the theorem on the continuous dependence of a solution on the initial conditions and by virtue of the continuity of function V there exists for the number $T>0$ and $N>0$ such that from $k_i>N$ follows

$$V(x(T;t_0,x(t_0+k_i\tau;t_0,x_0)))<V_* \tag{3.13}$$

From (3.11) and from the property of uniqueness of (2.6) follows:

$$x(T;t_0,x(t_0+k_i\tau;t_0,x_0))=x(T+k_i\tau;t_0+k_i\tau,x(t_0+k_i\tau;t_0,x_0))=x(T+k_i\tau;t_0,x_0) \tag{3.14}$$

Substituting (3.14) into (3.13) we obtain $V(x(T+k_i\tau;\ t_0,\ x_0))<V_*$, which contradicts

(3.12). Thus, $V_* = 0$, which is what we had to prove.

Note. It can be shown [15] that when the hypotheses of Theorem 25 are fulfilled the asymptotic y-stability is uniform in $\{t_0, x_0\}$.

Theorem 26 [13]. If a function $V(x)$ is such that $V(x) \geqslant a(\|y\|)$, while $V^{\cdot}$ satisfies condition (2.29), and moreover the set $\{x: y = 0\}$ is invariant while $M \setminus \{x: y = 0\}$ does not contain entire trajectories, then the motion $x = 0$ is asymptotically y-stable.

The proof [13] is based on the properties of the ω-limit points of dynamic systems and is carried out according to the same plan as for the proofs of Theorems 11 and 12.

9. Theorem 27 [9]. If there exists a function $V(t, x)$ satisfying conditions (1.7) and (2.5) (in the sense of Definition 3) in the region (1.4) and if for any $t_0 \geqslant 0$, $\lambda > 0$ we can find $M(t_0, \lambda) > 0$ such that

$$\| Y(t, x) \| \leqslant M \quad \text{for} \quad (t, x) \in E(t_0, \lambda) \tag{3.15}$$

then the motion $x = 0$ is asymptotically y-stable in-the-large.

Proof. The conditions of item 1) of Theorem 1 are fulfilled, therefore we need to prove only that $\lim \| y(t; t_0, x_0) \| = 0$ as $t \to \infty$ for any x_0 and $t_0 \geqslant 0$. Assume the contrary: let there exist a number $l > 0$ and a sequence $t_k \to \infty$, $t_k - t_{k-1} \geqslant \alpha > 0$, $k = 1,2,3,\ldots$, such that $\| y(t_k; t_0, x_*) \| \geqslant l$ for some x_* and $t_0 \geqslant 0$. We denote $\| x_* \| = \lambda$; from (3.15) there follows the existence of a number β, $0 < \beta < \alpha/2$, such that $\| y(t; t_0, x_*) \| \geqslant l/2$ when $t \in [t_k - \beta, t_k + \beta]$, $k = 1, 2, 3, \ldots$. By virtue of (2.5), from (2.2) follows

$$0 \leqslant V(t_k + \beta; x(t_k + \beta; t_0, x_*)) \leqslant V(t_0, x_*) + \sum_{i=1}^{k} \int_{t_i - \beta}^{t_i + \beta} V^{\cdot}(\tau, x(\tau; t_0, x_*))\, d\tau \leqslant$$
$$\leqslant V(t_0, x_*) - 2k\beta c(l/2)$$

which is impossible for k sufficiently large. The theorem is proved.

Condition (3.15) permits us to relinquish the requirement of a strong infinitesimal upper bound and of an infinite lower bound [25] for the function V, however, in this case the uniformity of the asymptotic y-stability is not guaranteed. For comparison we offer the following generalization of the theorem appearing in [20, 36, 37, 29] on asymptotic stability in-the-large:

Theorem 28 (*). If a function V exists in region (1.4), satisfying conditions (3.2) and (3.3) and, moreover, if

$$V(t, x) \rightrightarrows +\infty \quad \text{as} \quad \sum_{i=1}^{k} x_i^2 \to +\infty \tag{3.16}$$

then the motion $x = 0$ is asymptotically y-stable in-the-large; here $\| y(t; t_0, x_0) \| \to 0$ as $t \to \infty$ uniformly in

$$\{t_0 \geqslant 0, (x_{10}, \ldots, x_{k0}) \in K, \ | x_{j0} | < +\infty \ (j = k+1, \ldots, n)\},$$

where K is an arbitrary compactum of the space $\{x_1, \ldots, x_k\}$, the solution of system (1.1) possesses property (R), and

$$X_i(t, 0, \ldots, 0, x_{k+1}, \ldots, x_n) \equiv 0 \qquad (i = 1, \ldots, k) \tag{3.17}$$

Proof. The asymptotic y-stability, property (R) and identity (3.17) follow from Theorems 16 and 17. Let compactum K be given. We denote

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*) This theorem has been proven by A. S. Oziraner.

$$b_0 = \max\left[b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right): \quad (x_1, \ldots, x_k) \in K\right]$$

By virtue of (3.16) there exists $R > 0$ such that

$$V(t, \mathbf{x}) > b_0 \quad \text{for } \sum_{i=1}^{k} x_i^2 > R^2$$

Consequently,

$$\sum_{i=1}^{K} x_i^2(t; t_0, \mathbf{x}_0) \leqslant R^2 \quad \text{for } t \geqslant t_0$$

if $t_0 \geqslant 0$, $(x_{10}, \ldots, x_{k0}) \in K$, $|x_{j0}| < \infty \quad (j = k+1, \ldots, n)$.

From the conditions imposed on the function V it follows (c. f. items 2) and 3) of Theorem 1) that for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$ if only

$$t_0 \geqslant 0, \quad \sum_{i=1}^{k} x_{i0}^2 < \delta^2, \quad |x_{j0}| < \infty \quad (j = k+1, \ldots, n)$$

We set $T(\varepsilon) = 2b_0 / c(\delta(\varepsilon))$ and we let $(x_{10}, \ldots, x_{k0}) \in K$. If we assume that

$$\sum_{i=1}^{k} x_i^2(t; t_0, \mathbf{x}_0) \geqslant \delta^2(\varepsilon) \quad \text{for } t \in (t_0, t_0 + T)$$

then from (2.2) we obtain

$$0 \leqslant V(t_0 + T, \mathbf{x}(t_0 + T; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) - c(\delta(\varepsilon))T \leqslant b_0 - c(\delta(\varepsilon))T < 0$$

which is impossible. Consequently, there exists an instant $t_* \in (t_0, t_0 + T)$ for which

$$\sum_{i=1}^{k} x_i^2(t_*; t_0, \mathbf{x}_0) < \delta^2(\varepsilon)$$

But then $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for $t \geqslant t_0 + T > t_*$. The theorem is proved.

Note. If in Theorems 25 and 26 we additionally require $a(\|\mathbf{y}\|) \to \infty$ as $\|\mathbf{y}\| \to \infty$, then the motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable in-the-large [13].

10. Let there be given the system of equations of perturbed motion of a controlled system

$$\mathbf{x} = \mathbf{X}(t, \mathbf{x}, \mathbf{u}) \qquad (\mathbf{u} = (u_1, \ldots, u_r)) \tag{3.18}$$

whose right-hand sides are defined and are continuous in the region

$$t \geqslant 0, \quad \|\mathbf{y}\| \leqslant H > 0, \quad 0 \leqslant \|\mathbf{z}\| < +\infty, \quad 0 \leqslant \|\mathbf{u}\| < +\infty \tag{3.19}$$

We seek $\mathbf{u}$ in the form $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$; here it is assumed that the function $\mathbf{u}(t, \mathbf{x})$ is defined and is continuous in region (1.2), while for $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ the system (3.18) satisfies restrictions imposed on system (1.1) and in addition

$$\mathbf{X}(t, 0, 0) \equiv 0, \qquad \mathbf{u}(t, 0) \equiv 0$$

The control performance index is taken to be the condition that the integral

$$J = \int_{t_0}^{\infty} \omega(t, \mathbf{x}[t], \mathbf{u}[t])\, dt$$

is minimum. Here $\omega(t, \mathbf{x}, \mathbf{u}) \geqslant 0$ is a scalar function continuous in region (3.19), $\mathbf{x}[t]$

is the solution of system (3.18) for $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, $\mathbf{u}[t] = \mathbf{u}(t, \mathbf{x}[t])$. The optimal $\mathbf{y}$-stabilization problem [10, 32] consists of finding the function $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$ which ensures the asymptotic $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$, where the inequality

$$\int_{t_0}^{\infty} \omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])\,dt \leqslant \int_{t_0}^{\infty} \omega(t, \mathbf{x}^*[t], \mathbf{u}^*[t])\,dt$$

for $t_0 \geqslant 0$, $\|\mathbf{x}[t_0]\| \leqslant \lambda$, $\lambda = \text{const} > 0$, must be satisfied for any other function $\mathbf{u} = \mathbf{u}^*(t, \mathbf{x})$ having the same property. We introduce the notation [35]

$$B[V, t, \mathbf{x}, \mathbf{u}] = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i(t, \mathbf{x}, \mathbf{u}) + \omega(t, \mathbf{x}, \mathbf{u}) \tag{3.20}$$

Theorem 29 [10, 32]. If there exist a function $V^\circ(t, \mathbf{x})$ satisfying inequalities (1.7) and (1.8) (respectively, inequalities (1.7) and (1.10)) and a function $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$ for which:

1) $W(t, \mathbf{x}) \equiv -\omega(t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x})) \leqslant -c(\|\mathbf{y}\|)$ (respectively, $W(t, \mathbf{x}) \equiv -\omega(t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x})) \leqslant -c(\|\mathbf{x}\|)$)

2) $B[V^\circ, t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x})] = 0$;

3) $B[V^\circ, t, \mathbf{x}, \mathbf{u}] \geqslant 0$ for any $\mathbf{u}$,

then the function $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$ solves the optimal $\mathbf{y}$-stabilization problem. Here

$$\int_{t_0}^{\infty} \omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t]), dt = \min \int_{t_0}^{\infty} \omega(t, \mathbf{x}[t], \mathbf{u}[t])\,dt = V^\circ(t_0, \mathbf{x}[t_0]) \tag{3.21}$$

Proof. From (3.20) and condition 2) of the theorem it follows that $V^{\circ\cdot} = W$ relative to the system $\mathbf{x}^\cdot = \mathbf{X}(t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x}))$. Therefore, all the hypotheses of Theorem 16 are fulfilled for $k = m$ (respectively, for $k = n$). Let us prove relation (3.21). By integrating the equality $dV^\circ(t, \mathbf{x}^\circ[t]) / dt = -\omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])$ and taking into account that $\lim V^\circ(t, \mathbf{x}^\circ[t]) = 0$ as $t \to \infty$, we obtain

$$V^\circ(t_0, \mathbf{x}^\circ[t_0]) = \int_{t_0}^{\infty} \omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])\,dt \tag{3.22}$$

By virtue of condition 3) of the theorem the inequality $dV^\circ(t, \mathbf{x}^*[t]) / dt \leqslant -\omega(t, \mathbf{x}^*[t])$ is valid for every function $\mathbf{u} = \mathbf{u}^*(t, \mathbf{x})$ ensuring the asymptotic $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$; by integrating this inequality and taking into account that $\lim V^\circ(t, \mathbf{x}^*[t]) = 0$ as $t \to \infty$ we obtain

$$V^\circ(t_0, \mathbf{x}^*[t_0]) \leqslant \int_{t_0}^{\infty} \omega(t, \mathbf{x}^*[t], \mathbf{u}^*[t])\,dt \tag{3.23}$$

($\mathbf{x}^*[t_0] = \mathbf{x}^\circ[t_0]$). Then, (3.21) follows from (3.22) and (3.23), which is what we had to prove.

This result generalizes a theorem appearing in [35], Supplement IV. One of the methods for solving the optimal stabilization problem was proposed in [32].

11. Only the first steps have been taken in the $\mathbf{y}$-stability problem in the linear approximation [11]. In this subsection we take $V^\cdot(t, \mathbf{x}) = \{D^+ V(\tau, \mathbf{x}(\tau; t, \mathbf{x}))\}_{\tau=t}$. At first we consider the linear system

$$\mathbf{y}^\cdot = A(t)\,\mathbf{y} + B(t)\,\mathbf{z}, \qquad \mathbf{z}^\cdot = C(t)\,\mathbf{y} + D(t)\,\mathbf{z} \tag{3.24}$$

in which A, B, C, D are matrix-valued functions of appropriate orders, continuous for $t \geqslant 0$. It is known that any solution $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (3.24) is defined for

$t \in [0, \infty)$.

Theorem 30 [11]. The motion $x = 0$ of system (3.24) is exponentially asymptotically y-stable if and only if there exists a function $V(t, x)$ satisfying the conditions

$$\|y\| \leqslant V(t,x) \leqslant M(\|y\| + \|z\|) \tag{3.25}$$

$$|V(t,x) - V(t,x')| \leqslant M(\|y - y'\| + \|z - z'\|) \tag{3.26}$$

$$V^{\cdot}(t,x) \leqslant -\alpha V(t,x) \tag{3.27}$$

Proof. 1) Sufficiency. Integrating (3.27) we obtain

$$V(t, x(t; t_0, x_0)) \leqslant V(t_0, x_0) \exp[-\alpha(t - t_0)] \tag{3.28}$$

By virtue of inequalities (3.25),

$$\|y(t; t_0, x_0)\| \leqslant V(t, x(t; t_0, x_0)) \tag{3.29}$$

$$V(t_0, x_0) \leqslant M(\|y_0\| + \|z_0\|) \tag{3.30}$$

Then, (1.5) follows from (3.28) – (3.30).

2) Necessity. Suppose that (1.5) holds. We set

$$V(t,x) = \sup[\|y(t+\tau; t, x)\| \exp(\alpha\tau) : \tau \geqslant 0] \tag{3.31}$$

Obviously, $V(t,x) \geqslant \|y\|$. By virtue of (1.5), from (3.31) it follows that

$$V(t,x) \leqslant M(\|y\| + \|z\|)$$

Since $x(t; t_0, x_0)$ depends linearly on x_0,

$$y(t+\tau; t, x) - y(t+\tau; t, x') = y(t+\tau; t, x - x')$$

Consequently,

$$|V(t,x) - V(t,x')| \leqslant \sup_{\tau \geqslant 0}[\|y(t+\tau; t, x - x')\| \exp(\alpha\tau)] \leqslant M(\|y - y'\| + \|z - z'\|)$$

Further, we have

$$V(t+h, x(t+h; t, x)) = \sup_{\tau \geqslant 0}[\|y(t+h+\tau; t+h, x(t+h; t, x))\| \exp(\alpha\tau)] =$$
$$= \sup_{\tau \geqslant 0}[\|y(t+h+\tau; t, x)\| \exp(\alpha\tau)] = \sup_{\tau \geqslant h}[\|y(t+\tau; t, x)\| \exp(\alpha\tau) \exp(-\alpha h)] \leqslant$$
$$\leqslant \sup_{\tau \geqslant 0}[\|y(t+\tau; t, x)\| \exp(\alpha\tau) \exp(-\alpha h)] = V(t,x) \exp(-\alpha h)$$

Consequently,

$$\frac{1}{h}[V(t+h, x(t+h; t, x)) - V(t,x)] \leqslant \frac{1}{h}[\exp(-\alpha h) - 1] V(t,x) \tag{3.32}$$

whence (3.27) follows as $h \to +0$. Note that the function $V(t, x)$ is continuous. Indeed,

$$|V(t+h, x') - V(t,x)| \leqslant |V(t+h, x') - V(t+h, x)| +$$
$$+ |V(t+h, x) - V(t+h, x(t+h; t, x))| + |V(t+h, x(t+h; t, x)) - V(t,x)|$$

The first two terms on the right-hand side tend to zero as $h \to 0, \|x - x'\| \to 0$ since V satisfies a Lipschitz condition, the last term tends to zero by virtue of (3.32). The theorem is proved.

We now consider the perturbed system

$$y^{\cdot} = A(t)y + B(t)z + f(t, y, z), \quad z^{\cdot} = C(t)y + D(t)z + g(t, y, z) \tag{3.33}$$

We assume that the functions f and g satisfy a Lipschitz condition in (y, z), $\|f(t,0,0)\| + \|g(t,0,0)\| \equiv 0$ and

$$\|\mathbf{f}(t, \mathbf{y}, \mathbf{z})\| + \|\mathbf{g}(t, \mathbf{y}, \mathbf{z})\| \leqslant \omega(t, \|\mathbf{y}\|) \tag{3.34}$$

moreover, the function $\omega(t, u)$ is continuous for $t \geqslant 0$, $u \geqslant 0$, locally satisfies a Lipschitz condition in u, and does not decrease with respect to u, $\omega(t, 0) = 0$. $V_{(1)}^{\cdot}$ and $V_{(2)}^{\cdot}$ be the derivatives of the function V relative to systems (3.24) and (3.33).

Theorem 31 [11]. Let the motion $\mathbf{x} = 0$ of system (3.24) be exponentially asymptotically $\mathbf{y}$-stable. If inequality (3.34) is fulfilled, then the $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$ of system (3.33) is of the same nature as the stability of the solution $u = 0$ of the comparison equation

$$u^{\cdot} = -\alpha u + M\,\omega(t, u) \tag{3.35}$$

Proof. By Theorem 30 there exists a function $V(t, \mathbf{x})$ satisfying the conditions (3.25) – (3.27). For this function

$$V_{(2)}^{\cdot}(t, \mathbf{x}) \leqslant V_{(1)}^{\cdot}(t, \mathbf{x}) + M(\|\mathbf{f}\| + \|\mathbf{g}\|) \leqslant -\alpha V(t, \mathbf{x}) + M\omega(t, \|\mathbf{y}\|)$$

whence, making use of the inequality $V \geqslant \|y\|$, we obtain

$$V_{(2)}^{\cdot}(t, \mathbf{x}) \leqslant -\alpha V(t, \mathbf{x}) + M\omega(t, V(t, \mathbf{x}))$$

Consequently, $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant u(t; t_0, u_0)$ follows from $V(t_0, \mathbf{x}_0) \leqslant u_0$ whence $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \leqslant u(t; t_0, u_0)$, which proves the theorem.

For example, if the solution $u = 0$ of Eq. (3.35) is exponentially asymptotically stable, then the motion $\mathbf{x} = 0$ of system (3.33) is exponentially asymptotically $\mathbf{y}$-stable.

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